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Semiclassical Cosmological Perturbations Generated During Inflation

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Interaction with the environment may induce stochastic semiclassical dynamics in open quantum systems. In the gravitational context, stress-energy fluctuations of quantum matter fields give rise to a stochastic behavior in the spacetime geometry. The Einstein–Langevin equation is a suitable tool to take these effects into account when addressing the backreaction problem in semiclassical gravity. We analyze within this framework the generation of gravitational fluctuations during inflation, which are of great interest for large-scale structure formation in cosmology.

1. INTRODUCTION

One of the key problems in modern cosmology is that of cosmic structure formation [1, 2]. If an inflationary period is present, the initial seeds for structure formation are supposed to be originated by the quantum fluctuations of the inflaton field, which is responsible for driving inflation [3]. By semiclassical backreaction on the spacetime geometry, these quantum fluctuations will, in turn, produce fluctuations on the spacetime metric. Here we want to look at this problem within the context of a simple chaotic inflationary model by means of a recently suggested formalism. In this formalism classical metric fluctuations induced by quantum matter fluctuations are described by a Langevin-type equation [4]. This is an alternative to the more usual approach in which some perturbative degrees of freedom of the gravitational field are also quantized [5].

The idea behind this approach is to relate the backreaction problem in semiclassical gravity with the dynamics of open quantum systems. In fact,

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there are a number of situations in which one is interested in the observables and the dynamics of a few degrees of freedom from a whole closed quantum system undergoing unitary evolution. These degrees of freedom constitute an open system whose dynamics is no longer unitary due to its interaction with the remaining degrees of freedom of the whole system, which constitute the environment [6, 7].

For the existence of a semiclassical regime for the system dynamics two requirements are needed [8, 9]. The first is decoherence, which guarantees that probabilities can be consistently assigned to histories describing the evolution of the system. The second is that these probabilities should be peaked near histories which correspond to solutions of classical equations of motion. The effect of the environment plays a crucial role in the semiclassical dynamics of the system. In fact, on one hand, it may provide enough induced decoherence through the entanglement between system and environment [10, 11, 7]. On the other hand, the environment backreaction on the system dynamics will produce both dissipation and noise (commonly connected by fluctuation-dissipation relations). The environment may thus induce a semiclassical stochastic dynamics on the system, which may be suitably described by a Langevin-type equation [9].

The plan of the paper is the following: In Section 2 we give a brief summary of the Einstein–Langevin equation. We apply this formalism in Section 3 to study the generation of cosmological gravitational perturbations during inflation by considering the simplest model leading to chaotic inflation. We finally discuss our main conclusions in Section 4. Throughout the paper we use natural units ($\hbar = c = 1$) and the $(+, +, +)$ sign convention of ref. 12.

2. EINSTEIN–LANGEVIN EQUATION

In the context of semiclassical gravity one treats the matter fields as quantum fields on a classical curved spacetime. As a consequence of their energy density, these fields act as gravitational sources which modify the spacetime geometry. To study this backreaction effect one usually uses the so-called semiclassical Einstein equation

$$G_{ab}[g] = \frac{8\pi}{m_p^2} \langle T_{ab}[g, \hat{\phi}[g]] \rangle_{\text{ren}} \tag{1}$$

where the renormalized expectation value of the stress tensor operators of the quantum matter fields in some quantum state are introduced as gravitational sources. There are, however, some situations in which the fluctuations of the stress tensor operator are important [13]. In those cases we cannot expect that the semiclassical Einstein equation provides the actual dynamics of the spacetime metric any longer, but some kind of *averaged* description.

It may be useful to consider the spacetime metric as an open system which interacts gravitationally with the quantum matter fields, which constitute the environment [14, 15]. In this case the system will exhibit a stochastic dynamics with fluctuations due to the noise induced by the environment. In order to take this effect into account, the following modified equation, known as the Einstein–Langevin equation, has been suggested [4]:

$$G_{ab}[g + h] - \frac{8\pi}{m_p^2} \langle \hat{T}_{ab}[g + h] \rangle_{\rm ren} = \frac{8\pi}{m_p^2} \xi_{ab}[g] \qquad (2)$$

where g is a solution of the semiclassical Einstein equation which is used as the background metric, and h is a linear perturbation. The field $\zeta_{ab}[g]$ is a Gaussian stochastic classical source with the following properties:

$$\langle \xi_{ab}(x) \rangle_\xi = 0 \qquad (3)$$

$$\langle \xi_{ab}(x)\xi_{cd}(y) \rangle_\xi = \tfrac{1}{2}\langle \{ \hat{t}_{ab}(x), \hat{t}_{cd}(y) \} \rangle [g] \qquad (4)$$

where $\hat{t}_{ab}(x) \equiv \hat{T}_{ab}(x) - \langle \hat{T}_{ab}(x) \rangle$. We use the two different notations, $\langle \cdot \rangle_\xi$ and $\langle \cdot \rangle$, to explicitly distinguish the average associated to a classical stochastic process from the expectation value of quantum operators. The correlation function for the stochastic source, which will generate a stochastic dynamics on the spacetime geometry, was precisely chosen to take into account the quantum fluctuations of the stress tensor.

3. COSMOLOGICAL PERTURBATIONS GENERATED DURING INFLATION

Let us now consider the simplest model leading to chaotic inflation [3], which is driven by a massive real scalar field $\hat{\phi}$ minimally coupled to the spacetime curvature (this field is usually called the *inflaton*). The corresponding Lagrangian density is thus

$$\mathscr{L}(\hat{\phi}) = \tfrac{1}{2} g^{ab} \nabla_a \hat{\phi} \nabla_b \hat{\phi} + \tfrac{1}{2} m^2 \hat{\phi}^2 \qquad (5)$$

A few comments are in order. First, the condition for the existence of an inflationary period (characterized by an accelerated expansion of spacetime) is that the value of the field averaged over a region with a typical size equal to the Hubble radius (the so-called horizon scale) is higher than the Planck mass m_p. In fact, in order to have enough inflation to solve the horizon and the flatness problem, more than 60 e-folds are needed. To achieve that, the scalar field should begin with a value higher than $3m_p$. On the other hand, as will be shown below, the small value of the cosmic microwave background (CMB) large-scale anisotropies measured by COBE [16] imposes a severe constraint on the inflaton mass m, which should be of the order of $10^{-6} m_p$.

We want to study small metric perturbations around a Robertson–Walker geometry. For this purpose we need to deal with the corresponding gauge freedom either by choosing a particular gauge or by working with gauge-invariant quantities [5]. We will restrict our study to scalar-type perturbations of the metric. The expression for the perturbed metric in the longitudinal gauge is then

$$ds^2 = a^2(\eta)\{-[1 + 2\Phi(x)]d\eta^2 + [1 - 2\Psi(x)]\delta_{ij}\, dx^i\, dx^j\} \tag{6}$$

where the two functions $\Phi(x)$ and $\Psi(x)$ correspond in this case to Bardeen's gauge-invariant variables and $a^2(\eta)$ is the cosmological scale factor of the background Robertson–Walker geometry. As shown below, the Einstein–Langevin equation (2) is gauge invariant. Therefore, we can work in a given gauge and finally extract the desired gauge-invariant quantities in a consistent way. To see how the first term of Eq. (2) is gauge invariant, one uses the following result for linear perturbations in h:

- $A^a_b[g + h]$ is gauge invariant if and only if $\mathcal{L}_{\vec{\varsigma}}(A^a_b[g]) = 0$ for any vector field $\vec{\varsigma}(x)$ and this is equivalent to $A^a_b[g] \propto \delta^a_b$ (zero being a particular case).

The left hand side of our Einstein–Langevin equation is thus gauge invariant if $G^{(0)}_{ab}[g] - (8\pi/m_p^2)\langle \hat{T}^{[0]}_{ab}[g]\rangle_{\text{ren}} = 0$, but this is indeed the case since the background metric g is taken to be a solution of the semiclassical Einstein equation. On the other hand, the right hand side of Eq. (2) is explicitly gauge invariant since it does not depend on the perturbed metric.

It is convenient to decompose the inflaton scalar field in the following way: $\hat{\phi}(x) = \phi(t) + \hat{\varphi}(x)$, where $\phi(t)$ is the homogeneous background solution, which is compatible with the background metric through the semiclassical Einstein equation, and $\hat{\phi}(x)$ corresponds to a free massive quantum scalar field with zero expectation value on the spacetime with the background metric: $\langle\hat{\phi}(x)\rangle_g = 0$. The two main ingredients that we need for our Einstein–Langevin equation are the renormalized expectation value of the stress tensor on the spacetime with the perturbed metric $\tilde{g} = g + h$ and the noise kernel, which takes into account the fluctuations of the stress tensor evaluated on the background metric. The stress tensor of a minimally coupled massive scalar field is

$$\hat{T}_{\mu\nu} = \partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - \tfrac{1}{2}\tilde{g}_{\mu\nu}(\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} + m^2\hat{\phi}^2) \tag{7}$$

Using the decomposition for the scalar field introduced above, we rewrite the renormalized expectation value for the stress tensor as

$$\langle\hat{T}_{\mu\nu}[g + h]\rangle^{ren} = \langle\hat{T}_{\mu\nu}[g + h]\rangle_{\phi\phi} + \langle\hat{T}_{\mu\nu}[g + h]\rangle_{\phi\varphi} + \langle\hat{T}_{\mu\nu}[g + h]\rangle^{\text{ren}}_{\varphi\varphi} \tag{8}$$

where only the homogeneous solution for the scalar field contributes to the first term. The second term is proportional to $\langle\hat{\varphi}[g + h]\rangle$, but this quantity is no longer zero since the field dynamics is considered on the perturbed spacetime. Finally, the last term corresponds to the expectation value of the stress tensor for a free scalar field on a spacetime with the perturbed metric. In the usual approach when computing fluctuations during inflation, $\hat{\varphi}$ is treated perturbatively. This last term, being quadratic in $\hat{\varphi}$, is of higher order and will not be taken into account.

As for the noise kernel, after using the previous decomposition, the following expression is obtained:

$$\langle\{\hat{t}_{\mu\nu}, \hat{t}_{\rho\sigma}\}\rangle[g] = \langle\{\hat{t}_{\mu\nu}, \hat{t}_{\rho\sigma}\}\rangle_{\phi\varphi}[g] + \langle\{\hat{t}_{\mu\nu}, \hat{t}_{\rho\sigma}\}\rangle_{\varphi\varphi}[g] \tag{9}$$

where we have used the fact that $\langle\hat{\varphi}\rangle_g = 0 = \langle\hat{\varphi}\hat{\varphi}\hat{\varphi}\rangle_g$ for Gaussian states (those considered here) on the background geometry. It is important to note that both contributions to the noise kernel (the first term is quadratic in $\hat{\varphi}$ and the second one is quartic) are "conserved" separately since both $\phi(t)$ and $\hat{\varphi}(x)$ satisfy the Klein–Gordon equation on the background geometry. Due to this fact, the two corresponding stochastic sources can be consistently considered in an independent way. We are thus allowed to concentrate on the source associated to the first term from now on. The contribution of a term of the same sort as the second one has been discussed elsewhere [17]. One can check that the *space–space* components coming from the stress-tensor expectation value terms that we are considering and the stochastic source are diagonal, i.e., $\langle\hat{T}_{ij}\rangle = 0 = \xi_{ij}$ for $i \neq j$. This, in turn, implies that the two gauge-invariant quantities used to characterize the scalar-type metric perturbations must be equal: $\Phi = \Psi$ [5].

Let us write the Einstein–Langevin equation in Fourier space and consider the $0i$ component:

$$2ik_i(\mathcal{H}\Phi_k + \Phi_k') = \frac{8\pi}{m_p^2}\,\xi_{k0_i} \tag{10}$$

where k_i is the comoving momentum component associated to the comoving coordinate x^i (throughout the paper we use the index k to denote the comoving momentum vector $\vec{k}$ that labels the Fourier modes in flat space), primes denote derivatives with respect to the conformal time η, and $\mathcal{H} \equiv a'(\eta)/a(\eta)$. The left hand side is just the linearized Einstein tensor for the perturbed metric (6) [5]. There should also appear a nonlocal term of *dissipative* character coming from the second term in (8), which we have not considered in this work, where we are mainly concerned about the fluctuating part.

From this equation we may obtain the metric perturbations Φ_k in terms of the stochastic source $\xi_{k\,0_i}$. For this purpose we need the retarded propagator for the gravitational potential Φ_k, i.e., the required Green function to solve the inhomogeneous first-order differential equation (10) with the appropriate boundary conditions:

$$\tilde{G}_k^{\mathrm{ret}}(\eta, \eta') = -i\,\frac{4\pi}{k_i m_p^2}\left(\theta(\eta-\eta')\,\frac{a(\eta')}{a(\eta)} + f(\eta,\eta')\right) \tag{11}$$

where $f(\eta, \eta')$ is a homogeneous solution related to the chosen initial conditions. If we take, for instance, $f(\eta, \eta') = -\theta(\eta_0 - \eta')a(\eta')/a(\eta)$, we would obtain the stochastic evolution of the metric perturbations for $\eta > \eta_0$ due to the effect of the stochastic source after η_0. The correlation function for the metric perturbations is then given by the following expression:

$$\begin{aligned}\langle\Phi_k(\eta)\Phi_{k'}(\eta')\rangle_\xi = (2\pi)^3\delta(\vec{k}+\vec{k}')\int^{\eta} d\eta_1 \int^{\eta'} d\eta_2\, \tilde{G}_k^{\mathrm{ret}}(\eta,\eta_1)\tilde{G}_{k'}^{\mathrm{ret}}(\eta',\eta_2)\\ \times\langle\xi_{k0_i}(\eta_1)\xi_{k'0_i}(\eta_2)\rangle_\xi\end{aligned} \tag{12}$$

The correlation function for the stochastic source is, in turn, connected with the stress-energy fluctuations:

$$\begin{aligned}\langle\xi_{k0_i}(\eta_1)\xi_{-k0_i}(\eta_2)\rangle_\xi &= \tfrac{1}{2}\langle\{\hat{t}^{\,k}_{0_i}(\eta_1), \hat{t}^{\,-k}_{0_i}(\eta_2)\}\rangle_{\phi\varphi}\\ &= \tfrac{1}{2}k_i k_i \phi'(\eta_1)\phi'(\eta_2)G_k^{(1)}(\eta_1,\eta_2)\end{aligned} \tag{13}$$

where $G_k^{(1)}(\eta_1, \eta_2) = \langle\{\hat{\varphi}_k(\eta_1), \hat{\varphi}_{-k}(\eta_2)\}\rangle$ is the k-mode Hadamard function for a free minimally coupled scalar field which is in a state close to the Euclidean vacuum on an almost de Sitter background.

The so-called "slow-roll" parameters account for the fact that the background geometry is not exactly that of de Sitter spacetime [for which $a(\eta) = -1/H\eta$ with $-\infty < \eta < 0$]. It is also useful to compute the Hadamard function for a massless field and consider a perturbative expansion in terms of the dimensionless parameter m/m_p, for which observations seem to imply, as will be seen below, a value of the order of 10^{-6}. Thus, we will consider $\overline{G}_k^{(1)}(\eta_1, \eta_2) = a(\eta_1)a(\eta_2)G_k^{(1)}(\eta_1, \eta_2) = \langle 0|\{\hat{y}_k(\eta_1), \hat{y}_{-k}(\eta_2)\}|0\rangle$ such that $\hat{a}_k|0\rangle = 0$ with $\hat{y}_k(\eta) = a(\eta)\hat{\varphi}_k(\eta) = \hat{a}_k u_k(\eta) + \hat{a}^\dagger_{-k}u^*_{-k}(\eta)$ and $u_k(\eta) = (2k)^{-1/2}e^{-ik\eta}(1 - i/k\eta)$ corresponding to the positive-frequency k-mode for a massless minimally coupled scalar field in the Euclidean vacuum state on a de Sitter background [18].

The result to lowest order in the mass m of the inflaton field and the "slow-roll" parameters is

$$\begin{aligned}\langle\Phi_k(\eta)\Phi_{k'}(\eta')\rangle_\xi = \frac{64\pi^5}{m_p^4}\,(a(\eta)a(\eta'))^{-1}\,\delta(\vec{k}+\vec{k}')\\ \times\int_{\eta_0}^{\eta} d\eta_1\int_{\eta_0}^{\eta'} d\eta_2\; a(\eta_1)a(\eta_2)\;\dot{\phi}(\eta_1)\dot{\phi}(\eta_2)\overline{G}_k^{(1)}(\eta_1\eta_2)\end{aligned}$$

$$= 64\pi^5 \left(\frac{m}{m_p}\right)^2 k^{-3}\, \delta(\vec{k} + \vec{k}') \int_{k\eta_0}^{k\eta} d(k\eta_1) \int_{k\eta_0}^{k\eta'} d(k\eta_2)\, \frac{k\eta}{k\eta_1} \frac{k\eta'}{k\eta_2}$$

$$\times \left[\cos k(\eta_1 - \eta_2) \cdot \left(1 + \frac{1}{k\eta_1\, k\eta_2}\right) - \sin k(\eta_1 - \eta_2)\right.$$

$$\left. \cdot \left(\frac{1}{k\eta_1} - \frac{1}{k\eta_2}\right)\right]$$

$$= 64\pi^5 \left(\frac{m}{m_p}\right)^2 k^{-3}\, \delta(\vec{k} + \vec{k}') \left[\cos k(\eta - \eta')\right.$$

$$- \frac{1}{k\eta_0} (k\eta \cos k(\eta - \eta_0)$$

$$\left. + k\eta' \cos k(\eta' - \eta_0)) + \frac{k\eta k\eta'}{(k\eta_0)^2}\right] \quad (14)$$

where we used the lowest order approximation for $\dot{\phi}(t)$ during "slow roll" (overdots denote derivatives with respect to the physical time t): $\dot{\phi}(t) \simeq -m_p^2(m/m_p)$. We considered the effect of the stochastic source after the conformal time η_0. Notice that the result (14) is rather independent of the value of η_0 provided that it is negative enough, i.e., it corresonds to an early enough initial time. This weak dependence on the initial conditions is rather usual in this context and can be qualitatively understood: after a sufficient amount of time, the accelerated expansion for the quasi-de Sitter spacetime during inflation effectively erases any information about the initial conditions, which is redshifted away. The actual result will therefore be very close to that for $\eta_0 = -\infty$:

$$\langle \Phi_k(\eta) \Phi_{k'}(\eta') \rangle_\xi = 64\pi^5 \left(\frac{m}{m_p}\right)^2 k^{-3}\, (2\pi)^3\, \delta(\vec{k} + \vec{k}') \cos k(\eta - \eta') \quad (15)$$

4. CONCLUSIONS

It is of major interest to study the cosmological implications which can be extracted from our work, especially those related to large-scale gravitational fluctuations. These fluctuations are believed to play a crucial role in the generation of the large-scale structure and matter distribution observed in our present universe [1]. They are also tightly connected with the anisotropies in the CMB radiation, which decoupled from matter about 3×10^5 years after the Big Bang and provide us with very valuable information about the early universe [2].

From the analysis of our final result in Eq. (15) two main facts can be concluded. First, an almost Harrison–Zel'dovich scale-invariant spectrum seems to be obtained for large scales (small values of k). Second, no significant relaxation of the coupling parameter is found. Since we get $\langle \Phi_k(\eta)\Phi_{k'}(\eta')\rangle_\xi \propto (m/m_p)^2$ in agreement with the usual results [5, 19], the small value of the CMB anisotropies detected by COBE imposes a severe bound on the gravitational fluctuations, characterized by $\langle \Phi_k(\eta)\Phi_{k'}(\eta')\rangle_\xi$, which implies $(m/m_p) \sim 10^{-6}$, whereas the mechanisms considered in those works [20] which allowed an important relaxation of this fine tuning (due to the extremely homogeneous classical initial conditions taken for the inflaton field) resulted in $\langle \Phi_k(\eta)\Phi_{k'}(\eta')\rangle_\xi \propto (m/m_p)$.

It can be shown that genuine quantum correlation functions can be equivalently obtained through a stochastic description based on Langevin-type equations even in regimes where the actual dynamics of the system does not admit a description in classical terms [21]. The case of gravitational perturbations coupled to a scalar field is more subtle due to the existing gauge symmetry associated to diffeomorphic transformations and the subsequent constraints arising in the dynamics of the whole system. Nevertheless, total agreement with the purely quantum treatment [5] is expected at least for the case in which both gravitational inhomogenities and the scalar field are treated perturbatively to linear order [22].

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